Lesson 3-8 Inverses of Functions

Vocabulary

inverse of a function
identity function

▶ **BIG IDEA** From a function's description as a set of ordered pairs, by an equation, or by a graph, its inverse can be similarly described.

Mental Math

What operation undoes each action?

a. adding $\frac{2}{3}$ to a number

b. multiplying a number by $\frac{\pi}{2}$

c. squaring a positive number

The volume of a cube is $V = e^3$, where e is the length of an edge. Considering V as a function of e, we can write $V = f(e)$, and f is the *cubing* function. For example, the volume of a cube with 7-cm edges is $f(7) = 7^3 = 343 \text{ cm}^3$. However, sometimes you know the volume of a cube and need to find the edge length. Suppose a cube has a volume of 250 cm^3. Solve for the edge length by finding the cube root of 250; $e = \sqrt[3]{250} \approx 6.3$ cm. In general, $e = \sqrt[3]{V}$. Considering e as a function of V, we can write $e = g(V) = \sqrt[3]{V}$. The functions $f: e \rightarrow e^3$ and $g: V \rightarrow \sqrt[3]{V}$ are examples of *inverse functions*.

Finding the Inverse of a Function

Recall that a function can be considered as a set of ordered pairs in which each first element is paired with exactly one second element. If you switch coordinates in the pairs, the resulting set of ordered pairs is called the **inverse of the function**.

Example 1

Let $f = \{(-3, -5), (-2, 0), (-1, 3), (0, 4), (1, 3), (2, 0), (3, -5)\}$. Describe the inverse of f. Is the inverse a function?

Solution

Let g be the inverse of f. The ordered pairs in g are found by switching the x- and y-coordinates of each pair in f.

$g = \{(-5, -3), (0, -2), (3, -1), (4, 0), (3, 1), (0, 2), (-5, 3)\}$

The inverse is not a function because there are ordered pairs in which the same first element is paired with different second elements, such as $(-5, -3)$ and $(-5, 3)$.

If the original function is described by an equation, then switching the variables in the equation gives an equation for its inverse. This has the same effect as switching the coordinates of every ordered pair.

Example 2

a. **Give an equation for the inverse of the function described by $y = -x^2 + 4$.**

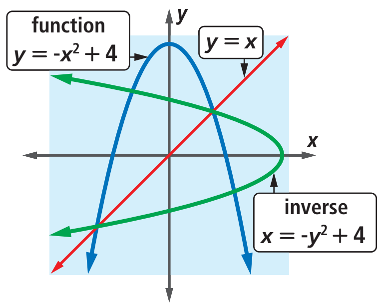
b. **Sketch a graph of $y = -x^2 + 4$ and the inverse on the same set of axes.**

c. **Is the inverse a function?**

Solution

a. To form an equation for the inverse, switch x and y. The inverse is described by the equation $x = -y^2 + 4$.

b. The graphs of $y = -x^2 + 4$ and its inverse, $x = -y^2 + 4$, are drawn at the right.



c. The graph of the inverse contains ordered pairs with the same first coordinate but different second coordinate and fails the vertical-line test. So $x = -y^2 + 4$ is not an equation of a function.

At the right are tables with some ordered pairs of each relation in Example 2. Notice that the ordered pairs, from the function to the inverse, follow the mapping $(x, y) \rightarrow (y, x)$. This mapping can be seen graphically as a reflection over the line $y = x$. So, the graphs of a function and its inverse are reflection images of each other over the line $y = x$.

function $y = -x^2 + 4$

x	y
-2	0
-1	3
0	4
1	3
2	0
3	-5
4	-12

inverse $x = -y^2 + 4$

x	y
0	-2
3	-1
4	0
3	1
0	2
-5	3
-12	4

STOP **QY1**

Activity

Step 1 Let $a(x) = \frac{1}{x - 3} + 4$. Describe in words what the function does to a number x according to the order of operations.

Step 2 Now describe in words how to "undo" the process you described in Step 1. Call this function b and write a formula for $b(x)$.

Step 3 Check your answers to Steps 1 and 2 by choosing a value for x, inputting that x-value into the formula for $a(x)$, and then substituting that output into the formula for $b(x)$. What was your result?

▸ QY1

If (3,12.5) is a point on the graph of a function, what point is on the graph of its inverse?

The Activity shows that when one function undoes the effects of another function, the original input x results. When two functions are comprised of the operations of each other in reverse order, they are inverse functions of each other. Sometimes it is more convenient to switch the x- and y-coordinates of the original function first and then perform the inverse operations to arrive at the inverse of the original function.

Example 3

Consider the function f with $f(x) = \frac{1}{x-3} + 4$.

a. Give an equation for the inverse of f.

b. Graph f and its inverse on the same set of axes.

Solution 1

a. Let $y = \frac{1}{x-3} + 4$. Switch x and y to find an equation for the inverse.

$$x = \frac{1}{y-3} + 4$$

This equation answers the question, but we usually solve for y.

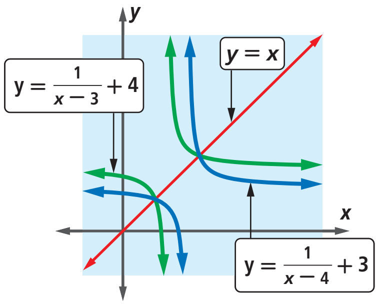
$$x - 4 = \frac{1}{y-3} \qquad \text{Subtract 4.}$$

$$y - 3 = \frac{1}{x-4} \qquad \text{Take reciprocals.}$$

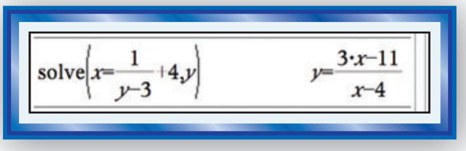
From this equation, by the Graph Translation Theorem, you can see that the graph of the inverse is the image of the graph of $y = \frac{1}{x}$ under the translation $(x, y) \to (x + 4, y + 3)$.

$$y = \frac{1}{x-4} + 3 \qquad \text{Add 3.}$$

b. The graphs of $y = \frac{1}{x-3} + 4$ and of $y = \frac{1}{x-4} + 3$ are shown at the right. To check that they are inverses, we also graphed $y = x$. Notice that each branch of the inverse is the image of one of the branches of the original hyperbola under a reflection over $y = x$.



Solution 2 Use the `solve` command on a CAS. Notice that the solution given here is in a different form than Solution 1. The CAS rewrote $\frac{1}{x-4} + 3$ using common denominators.



STOP **QY2**

Examples 2 and 3 demonstrate a significant feature of inverses. That is, not all inverses are functions. Looking at the graph and applying the definition of function reveals that the inverse from Example 2 is not a function, while the inverse from Example 3 is a function.

When the inverse of a function f is a function, it is denoted by the symbol f^{-1}, read "f inverse." With this notation, the rule for the inverse of f in Example 3 can be written $f^{-1}(x) = \frac{3x-11}{x-4}$. *Caution*: Note that f^{-1} does not denote the reciprocal of f, which is $\frac{1}{f}$.

STOP **QY3**

QY2

Show that $\frac{1}{x-4} + 3 = \frac{3x-11}{x-4}$.

QY3

a. What is the vertical asymptote of the graph of f in Example 3?

b. What is the vertical asymptote of the graph of f^{-1}?

Inverse Functions and Composition of Functions

Because the inverse of a function is found by switching the x- and y-coordinates, the domain and range of the inverse are found by switching the domain and range of the original function. Thus, the domain of f^{-1} is the range of f, and the range of f^{-1} is the domain of f. Hence, if f is a function whose inverse is also a function, $f(f^{-1}(x))$ and $f^{-1}(f(x))$ can always be calculated.

For example, for the function in Example 3,

$$f(f^{-1}(2)) = f\left(\frac{1}{2-4} + 3\right) = f(2.5) = \left(\frac{1}{2.5-3} + 4\right) = 2,$$

and $$f^{-1}(f(2)) = f^{-1}\left(\frac{1}{2-3} + 4\right) = f^{-1}(3) = \left(\frac{1}{3-4} + 3\right) = 2.$$

As you will see in Example 4, for these functions, $f \circ f^{-1}(x) = f^{-1} \circ f(x) = x$ for all values of x for which the composites are defined. This is why f^{-1} is called the inverse of f; f^{-1} undoes the effect of f. This property of the composition of inverses is an instance of the following theorem.

Inverses of Functions Theorem

Given any two functions f and g, f and g are inverse functions if and only if $f(g(x)) = x$ for all x in the domain of g, and $g(f(x)) = x$ for all x in the domain of f.

When f and g are inverse functions, then $f = g^{-1}$ and $g = f^{-1}$. The theorem states that two functions are inverses of each other if and only if $f \circ g$ and $g \circ f$ are the function I with $I(x) = x$, a function which is called an **identity function**. The Inverses of Functions Theorem enables you to test whether two functions are inverse functions even if you have not derived one from the other.

Example 4

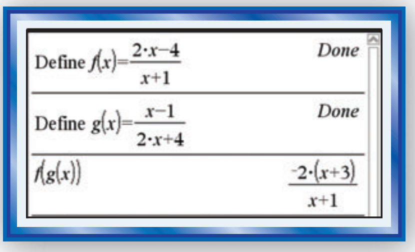
a. Use the Inverses of Functions Theorem to determine whether f and g, defined by $f(x) = \frac{2x-4}{x+1}$ and $g(x) = \frac{x-1}{2x+4}$, are inverses.

b. Verify your result in Part a by graphing f and g.

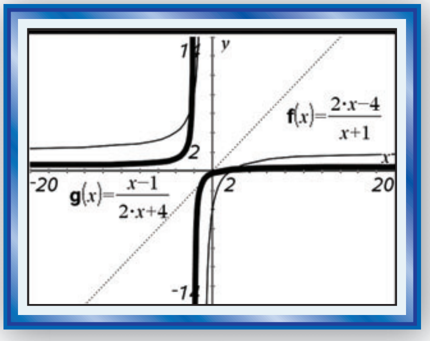
Solution

a. Define both relations on a CAS. Use the CAS to evaluate $f(g(x))$

Because $(f \circ g)(x) \neq x$, the functions are not inverses.



b. Graph f, g, and $y = x$. The graph of g is shown using a bold line. Note that the graphs of f and g are not reflection images of each other over $y = x$.



Questions

COVERING THE IDEAS

1. Define *inverse of a function*.

2. Let $f = \{(1, -4), (2, -6), (3, -8), (4, -10)\}$.
   a. Find g, the inverse of f.
   b. Graph f and g on the same set of axes.
   c. What transformation relates the graphs of f and g?

3. a. Suppose $f(x) = -2x^2$. Graph f and its inverse on the same axes.
   b. Explain why the inverse of f is not a function.

4. Give an example, different from those in the lesson, of a function whose inverse is not a function.

In 5–7, write an equation for the inverse of the function with the given equation. Solve your equation for y. Is the inverse a function?

5. $y = 2x - 4$
6. $y = -x^3$
7. $y = \sqrt{x}$

8. A rule for a function h is given. Is the inverse of h a function?
   a. $h(x) = |x + 2|$
   b. $h(x) = x + 2$

In 9 and 10, determine if f and g are inverses by finding $g \circ f(x)$ and $f \circ g(x)$. Then check your conclusion by graphing the functions.

9. $f(x) = x^3$; $g(x) = \sqrt[3]{x}$
10. $f(x) = \frac{2}{x} - 5$; $g(x) = \frac{2}{x + 5}$

APPLYING THE MATHEMATICS

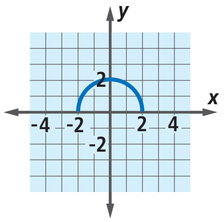
11. At one point in the summer of 2008, one U.S. dollar was worth 10.033 Mexican pesos. Let $M(x)$ be the cost in pesos of an item priced at x U.S. dollars and $U(x)$ be the cost in dollars of an item priced at x Mexican pesos.
   a. Write expressions for $M(x)$ and $U(x)$.
   b. What was the U.S. price of an item which cost 20,000 pesos?
   c. Are M and U inverses of each other? Justify your answer.

12. The rule for converting from degrees Fahrenheit to degrees Celsius is "subtract 32, then multiply by $\frac{5}{9}$."
   a. Determine the rule for converting Celsius to Fahrenheit.
   b. Do the two rules represent inverse functions? Why or why not?

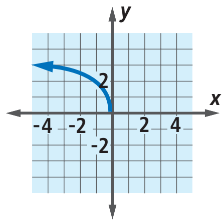
In 13 and 14, a graph is given.

   a. Sketch the graph of the inverse of the function.
   b. State whether or not the inverse is a function.

13.



14.



15. If h is the reciprocal function defined by $h(p) = \frac{1}{p}$, show that $h(p) = h^{-1}(p)$ for all $p \neq 0$. That is, h is its own inverse.

16. a. Let $f(x) = mx + b$, where $m \neq 0$. Find a formula for $f^{-1}(x)$.
   b. **True or False** The inverse of every function whose graph is a line is a function whose graph is a line. If false, give a counterexample.

17. An empty box weighs 11 ounces. It is filled with bolts weighing 8 oz each. The total weight in ounces of the box is given by $w(n) = 11 + 8n$ where n is the number of bolts. Write a formula for the inverse function which finds the number of bolts given the total weight.

REVIEW

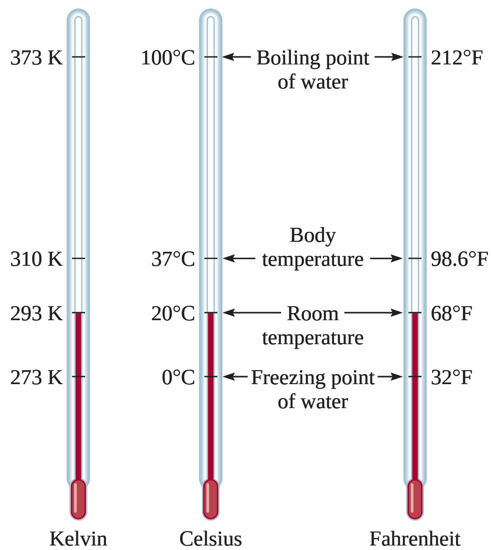
In 18 and 19, let $v(t) = 39t$ and $r(t) = t + 17$. (Lesson 3-7)

18. Evaluate each composite.
   a. $(r \circ v)(-1)$
   b. $r \circ r(-1)$

19. Find a formula for each function.
   a. $v \circ r$
   b. $v \circ v$

20. Consider the Fahrenheit to Celsius rule in Question 12 and the rule to convert from Celsius to Kelvin of "add 273.15." **(Lesson 3-7)**
   a. Write equations for functions that convert from Fahrenheit to Celsius and from Celsius to Kelvin.
   b. Find the composite function. What does this function do?

21. a. Prove that the function p defined by $p(x) = -|x| + 7$ is an even function.
   b. What type of symmetry does the graph of $y = p(x)$ have? **(Lesson 3-4)**



In 22 and 23, use the data in the table at the right that contains the number of cell phone subscribers (in thousands) and the average length of local calls.

	Subscribers	Min/Call
1990	5,283	2.20
1995	33,786	2.15
2000	109,478	2.56
2005	207,896	3.00

Source: CTIA - The Wireless Association

22. a. Graph the data for subscribers for a given year.
    b. Find the best exponential model relating the year to the number of subscribers.
    c. Use the model to predict the number of cell phone subscribers in 2010. **(Lesson 2-5)**

23. a. Graph the data for the average call length for a given year.
    b. Find a good model relating the year to the call length.
    c. Use the model to predict the average length of a cell phone call in 2010. **(Lesson 2-3)**

24. The table below contains average hourly earnings of production workers in the U.S. by month in 2006. **(Lesson 2-3)**
    The line of best fit for this data set is $y = 0.06x + 16.38$.

Month	1	2	3	4	5	6	7	8	9	10	11	12
Hourly Wage	16.43	16.49	16.55	16.63	16.66	16.73	16.79	16.84	16.88	16.94	16.99	17.07

Source: Bureau of Labor Statistics

    a. Use the line of best fit to predict the hourly wage for April 2008.
    b. Use a spreadsheet to find the residual and squared residual for each month.
    c. What is the sum of the squared residuals? What conclusion can you draw about that number?

25. Rewrite $p_1q_1 + p_2q_2 + \ldots + p_nq_n$ using Σ-notation. **(Lesson 1-2)**

EXPLORATION

26. The field of cryptology relies on inverses. Research the Caesar cipher to find out what it is and how it relates to inverse functions.

QY ANSWERS

1. (12.5, 3)
2. $\frac{1}{x-4} + 3$
   $= \frac{1}{x-4} + \frac{3(x-4)}{x-4} = \frac{1+3x-12}{x-4}$
   $= \frac{3x-11}{x-4}$
3. a. $x = 3$ b. $x = 4$